Certain results on entire functions defined by bicomplex Dirichlet series

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ABSTRACT

In this work, we have introduced and studied the Bicomplex version of Complex Dirichlet Series \( f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} \). We have derived condition for which the sum function of the Bicomplex Dirichlet Series \( f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi} \) represents an entire function. The Entireness of sum and Hadamard product of two Entire Bicomplex Dirichlet Series are also discussed.

Keywords: Dirichlet Series, Entire Dirichlet Series, Riemann Zeta Function, Hadamard Product

INTRODUCTION

Throughout this paper, the set of Bicomplex numbers is denoted by \( \mathbb{C}_2 \) and the sets of complex and real numbers are denoted by \( \mathbb{C}_1 \) and \( \mathbb{C}_0 \), respectively. For details of the theory of bicomplex numbers,\(^{1,2}\)

The set of Bicomplex Numbers defined as:
\[
\mathbb{C}_2 = \{ x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0, \\
i_1 \neq i_2 \text{ and } i_2^2 = -1, \ i_1 i_2 = i_2 i_1 \}
\]

We shall use the notations \( C(i_1) \) and \( C(i_2) \) for the following sets:
\[
C(i_1) = \{ u + i_1 v : u, v \in \mathbb{C}_0 \} \\
C(i_2) = \{ \alpha + i_2 \beta : \alpha, \beta \in \mathbb{C}_0 \}
\]

1.1 IDEMPOTENT ELEMENTS:

Besides 0 and 1, there are exactly two non – trivial idempotent elements in \( \mathbb{C}_2 \), denoted as \( e_1 \) and \( e_2 \) and defined as \( e_1 = \frac{1+i_1 i_2}{2} \) and \( e_2 = \frac{1-i_1 i_2}{2} \).

Note that \( e_1 + e_2 = 1 \) and \( e_1 e_2 = e_2 e_1 = 0 \).

1.2 CARTESIAN IDEMPOTENT SET:

Cartesian idempotent set \( X \) determined by \( X_1 \) and \( X_2 \) is denoted as \( X_1 \times_{e} X_2 \) and is defined as:
\[
X_1 \times_{e} X_2 = \{ (\xi, \eta) : \xi = \xi_1 e_1 + \xi_2 e_2, (\eta_1, \eta_2) \in X_1 \times X_2 \}
\]

\[
C_2 = C(i_1) \times_{e} C(i_2) = C(i_1) e_1 + C(i_2) e_2 \\
= \{ (\xi, \eta) : \xi = \xi_1 e_1 + \xi_2 e_2, (\eta_1, \eta_2) \in C(i_1) \times C(i_2) \}
\]

\[
C_2 = C(i_1) \times_{e} C(i_2) = C(i_1) e_1 + C(i_2) e_2 \\
= \{ (\xi, \eta) : \xi = \xi_1 e_1 + \xi_2 e_2, (\eta_1, \eta_2) \in C(i_1) \times C(i_2) \}
\]

1.3 IDEMPOTENT REPRESENTATION OF BICOMPLEX NUMBERS

(1) \( C(i_1) \)-idempotent representation of Bicomplex Number is given by
\[
\xi = z_1 + i_1 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2 \\
= i_1 \xi e_1 + \xi_2 e_2
\]
(II) C(i₂) - idempotent representation of Bicomplex Number is given by
\[ \xi = (x_1 + i_2 x_3) + i_1 (x_2 + i_2 x_4) = w_1 + i_1 w_2 \]
\[ = (w_1 - i_2 w_2) e_1 + (w_1 + i_2 w_2) e_2 = \xi_1 e_1 + \xi_2 e_2 \]

**Note 1.1:** Out of the two idempotent representation, we use C(i₁) - idempotent representation. All the results also proved with the help of C(i₂) - idempotent representation technique.

The norm in C₂ is defined as
\[ \|\xi\| = \left( |\xi_1|^2 + |\xi_2|^2 \right)^{1/2} = \left[ x_1^2 + x_2^2 + x_3^2 + x_4^2 \right]^{1/2} \]
\[ C_{\xi,\eta} \text{ becomes a modified Banach algebra, in the sense that} \]
\[ \|z\eta\| \leq \sqrt{2} \|z\| \|\eta\| \quad \text{... (1.1)} \]

### 1.4 Complex Dirichlet Series:4,

A Dirichlet series is a series of the form
\[ f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \text{ where } \{\lambda_n\} \text{ is a strictly monotonically increasing and unbounded sequence of positive real numbers, and } s = \sigma + \text{i}\tau \text{ is a complex variable.} \]

When the sequence \{\lambda_n\} of exponent is to be emphasized, such a series is called a complex Dirichlet series of type \lambda_n.4

A Dirichlet series of the type n is a power series in e^{-s} is given by
\[ f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} = \sum_{n=1}^{\infty} a_n (e^{-s})^n \]

A Dirichlet series of type \log n is the Generalized Riemann Zeta function is given by
\[ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \]

Abscissae of convergence and absolute convergence:
To every Dirichlet series, there exists a number \sigma_0 such that the Dirichlet series \[ f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \] converges for \Re(s) > \sigma_0 and diverges for \Re(s) < \sigma_0. The number \sigma_0 is called the abscissa of convergence of the series, and the line \Re(s) = \sigma_0 is called the line of convergence.

To every Dirichlet series, there exists a number \sigma such that the Dirichlet series \[ f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} \] is absolutely convergent for \Re(s) > \sigma, and not absolutely convergent for \Re(s) < \sigma (this region comprise the region \Re(s) < \sigma_0 of divergence, the region \sigma_0 < \Re(s) < \sigma of conditional convergence and the line \Re(s) = \sigma_0).

The quantity \sigma is called the abscissa of absolute convergence of the series \[ f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} \] and the line \Re(s) = \sigma is called the line of absolute convergence.

### 1.5 Entireness of Complex Dirichlet Series:

**Theorem 1.1:**

For the complex Dirichlet Series \[ \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \]
\[ \text{If } \lim_{n \to \infty} \frac{n}{\lambda_n} < \infty \text{, then } \sigma_0 = \sigma = \lim_{n \to \infty} \frac{\log |a_n|}{\lambda_n} \]

**Corollary 1.1:** For a Dirichlet Series \[ f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} \]
\[ \sigma_0 = \sigma = \lim_{n \to \infty} \frac{\log |a_n|}{n} \]

**Proof:**
\[ \lambda_n = n \Rightarrow \lim_{n \to \infty} \frac{n}{\lambda_n} = \lim_{n \to \infty} \frac{n}{n} = 1 < \infty \]
Hence, \[ \sigma_0 = \sigma = \lim_{n \to \infty} \frac{\log |a_n|}{n} \]

**Corollary 1.2:** The Complex Dirichlet Series \[ f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} \] represents an Entire function iff \[ |a_n|^{1/n} \to 0 \text{ as } n \to \infty \].

**Proof:**
\[ \lim_{n \to \infty} \frac{\log |a_n|}{n} = -\infty \Leftrightarrow \lim_{n \to \infty} \frac{\log |a_n|}{n} = -\infty \]
\[ \Leftrightarrow \lim \log |a_n|^{1/n} = -\infty \Leftrightarrow \lim |a_n|^{1/n} = 0 \]
\[ \Leftrightarrow |a_n|^{1/n} \to 0 \text{ as } n \to \infty \]
Hence \( f(s) = \sum_{n=1}^{\infty} a_n e^{-n^s} \) represents an Entire function if 
\[ |a_n|^{1/n} \to 0 \text{ as } n \to \infty. \]

2. BICOMPLEX DIRICHLET SERIES:

In this paper we discuss a Bicomplex Dirichlet Series of type n, which is a Bicomplex Power Series in \( e^{-\xi} \)
\[ f(\xi) = \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} = \sum_{n=1}^{\infty} a_n (e^{-\xi})^n \]
where \( \{a_n\} \) is a sequence of bicomplex numbers and \( \xi \) is a bicomplex variable.

Note that,
\[ \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} = \left[ \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \right] e_1 + \left[ \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \right] e_2 \]
\[ \Rightarrow f(\xi) = f(1, \xi) e_1 + 2f(2, \xi) e_2 \]
Where, \( f(\xi) = \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \) is a Bicomplex Dirichlet Series and
\[ 2f(2, \xi) = \sum_{n=1}^{\infty} 2a_n e^{-n^2 \xi} \]
are Complex Dirichlet Series.

Throughout, we denote the abscissae of convergence of the Complex Dirichlet series \( \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \) and \( \sum_{n=1}^{\infty} 2a_n e^{-n^2 \xi} \)
by \( \sigma_1 \) and \( \sigma_2 \) and abscissae of their absolute convergence by \( \overline{\sigma}_1 \) and \( \overline{\sigma}_2 \), respectively.

**Theorem 2.1:**

For the Bicomplex dirichlet series \( f(\xi) = \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \)
\[ \sigma_1 = \overline{\sigma}_1 = \sigma_2 = \overline{\sigma}_2 = \lim_{n \to \infty} \log \| a_n \|_n \]
if \( a_{1n} a_{4n} = a_{2n} a_{3n} \)

**Proof:**

\[ a_n = a_{1n} + i_1 a_{2n} + i_2 a_{3n} + i_1 i_2 a_{4n} \]
\[ a_n = (a_{1n} + a_{4n}) + i_1 (a_{2n} - a_{3n}) \]
Where, \( 1 \) \( a_n = (a_{1n} + a_{4n}) + i_1 (a_{2n} - a_{3n}) \)
and \( 2 \) \( a_n = (a_{1n} - a_{4n}) + i_1 (a_{2n} + a_{3n}) \)

\[ 1 |a_n|^2 = 2 |a_n|^2 = (a_{1n})^2 + (a_{2n})^2 + (a_{3n})^2 + (a_{4n})^2 \]
\[ \cdot a_{1n} a_{4n} = a_{2n} a_{3n} \]
Also,
\[ \|a_n\| = \sqrt{(a_{1n})^2 + (a_{2n})^2 + (a_{3n})^2 + (a_{4n})^2} \]
Hence, \( \|a_n\| = 1 |a_n| = 2 |a_n| \) if \( a_{1n} a_{4n} = a_{2n} a_{3n} \)

As, \( \sigma_1 = \overline{\sigma}_1 = \lim_{n \to \infty} \log \| a_n \|_n \)
and
\[ \sigma_2 = \overline{\sigma}_2 = \lim_{n \to \infty} \log \| a_n \|_n \]

**Theorem 2.2:**

The Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} a_n e^{-n^2 \xi} \) and the \( k^\text{th} \)
derivative defined by \( \sum_{n=1}^{\infty} (-n)^k a_n e^{-n^2 \xi} \) have the same region of convergence.

**Proof:**

\[ \sigma_1 = \overline{\sigma}_1 = \lim_{n \to \infty} \log \| a_n \|_n \]
and
\[ \sigma_2 = \overline{\sigma}_2 = \lim_{n \to \infty} \log \| a_n \|_n \]

Let \( \rho_1, \rho_2 \) and \( \overline{\rho}_1, \overline{\rho}_2 \) are the associated abscissae of convergence and absolute convergence of the Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} (-n)^k a_n e^{-n^2 \xi} \). The,
\[ \rho_1 = \overline{\rho}_1 = \lim_{n \to \infty} \log \| (-n)^k a_n \|_n \]
\[ = \lim_{n \to \infty} \log \| (-n)^k a_n \|_n \]
\[ = \lim_{n \to \infty} \log \| n^k a_n \|_n \]
\[ = \lim_{n \to \infty} \log \| n^k a_n \|_n \]

**Integrated Research Advances**

\[
= k \lim_{n \to \infty} \frac{\log n}{n} + \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} = 0 + \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n}
\]

\[
= \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} = \sigma_1
\]

Similarly,
\[
\rho_2 = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{n} = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{n} = \sigma_2.
\]

**Theorem 2.3:**

The Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} \alpha_n e^{-n \xi} \) and the Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} -\alpha_n e^{-n \xi} \) obtained after k-times term-by-term integration of \( \sum_{n=1}^{\infty} \alpha_n e^{-n \xi} \) have the same region of convergence.

**Proof:**

\[
\sigma_1 = \sigma_1 = \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} \quad \text{and} \quad \sigma_2 = \sigma_2 = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{n}
\]

Let \( \rho_1, \rho_2, \bar{\rho}_1, \bar{\rho}_2 \) are the associated abscissae of convergence and absolute convergence of the Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} \alpha_n e^{-n \xi} \).

Then, \( \rho_1 = \bar{\rho}_1 = \lim_{n \to \infty} \frac{\log^1 \alpha_n}{(-n)^k} \quad \text{and} \quad \rho_2 = \bar{\rho}_2 = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{(-n)^k} \)

\[
= \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} - k \lim_{n \to \infty} \frac{\log n}{n}
\]

\[
= \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} - k \lim_{n \to \infty} \frac{\log n}{n} = \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n} - 0
\]

\[
= \lim_{n \to \infty} \frac{\log^1 \alpha_n}{n}
\]

\[
= \sigma_1
\]

Similarly,
\[
\rho_2 = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{n} = \lim_{n \to \infty} \frac{\log^2 \alpha_n}{n} = \sigma_2.
\]

**Entire Bicomplex Dirichlet Series**

**Definition 2.1:**

The Bicomplex Dirichlet series \( f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n \xi} \) is said to be an entire Bicomplex Dirichlet Series if it is convergent in the entire \( \mathbb{C}_2 \)-space.

**Theorem 2.4:**

\[
\|\alpha_n\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty \quad \text{if and only if} \quad \|\alpha_n\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty,
\]

where \( \alpha_n = \alpha_n e_1 + \beta_n e_2 \).

**Proof:**

Let \( \|\alpha_n\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty \)

Given \( \varepsilon > 0 \), \( \exists \ m \in \mathbb{N} \), such that \( \|\alpha_n\|^{1/n} < \varepsilon \) \( \forall \ n \geq m \)

Now \( \forall \ n \geq m, \quad \|\alpha_n\| < \varepsilon \to \Rightarrow \quad \|\alpha_n\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty \)

Conversely let \( \|\alpha_n\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty \) and \( \varepsilon > 0 \), \( \exists \ m_1, m_2 \in \mathbb{N} \)

Such that \( \|\alpha_n\|^{1/n} < \varepsilon \) \( \forall \ n \geq m_1 \) and \( \|\alpha_n\|^{1/n} < \varepsilon \) \( \forall \ n \geq m_2 \)
Let \( m = \max (m_1, m_2) \)

Then \( \forall \ n \geq m \)

\[
\left| \alpha_n \right|^2 < \varepsilon \quad \text{and} \quad \left| \beta_n \right|^2 < \varepsilon
\]

\[
\Rightarrow \frac{1}{2} \left( \left| \alpha_n \right|^2 + \left| \beta_n \right|^2 \right) \leq \varepsilon
\]

\[
\Rightarrow 2 \left| \alpha_n \right|^2 < \varepsilon^2 + \varepsilon^2 = 2 \varepsilon^2
\]

\[
\Rightarrow \left\| \alpha_n \right\|^2 \to 0 \text{ as } n \to \infty
\]

**Theorem 2.5:**

The Bicomplex Dirichlet series \( f(\xi) = \sum \alpha_n e^{-n\xi} \) is an entire Bicomplex Dirichlet series if and only if both

\[
1 \left( f(\xi) \right) = \sum_1 \alpha_n e^{-n\xi} \quad \text{and} \quad 2 \left( f(\xi) \right) = \sum_2 \alpha_n e^{-n\xi}
\]

are entire Complex Dirichlet series.

**Corollary 2.1:** The Bicomplex Dirichlet series \( f(\xi) = \sum_1 \alpha_n e^{-n\xi} \) is an entire Bicomplex Dirichlet series if and only if \( \sigma_1 = -\infty \) and \( \sigma_2 = -\infty \).

**Corollary 2.2:** The Bicomplex Dirichlet series \( f(\xi) = \sum_1 \alpha_n e^{-n\xi} \) is an entire Bicomplex Dirichlet series if and only if \( \left\| \alpha_n \right\|^2 \to 0 \) and \( \left\| \beta_n \right\|^2 \to 0 \).

**Corollary 2.3:** The Bicomplex Dirichlet series \( f(\xi) = \sum_1 \alpha_n e^{-n\xi} \) is an entire Bicomplex Dirichlet series if \( \left\| \alpha_n \right\|^2 \to 0 \).

**Theorem 2.6:**

Let \( h(\xi) = \sum \alpha_n \beta_n e^{-n\xi} \) be the Hadamard product of \( f(\xi) = \sum \alpha_n e^{-n\xi} \) and \( g(\xi) = \sum \beta_n e^{-n\xi} \). If \( f \) and \( g \) are entire Bicomplex Dirichlet series, then \( h \) is also an entire Bicomplex Dirichlet series.

**Proof:**

\[
f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi} \quad \text{and} \quad g(\xi) = \sum_{n=1}^{\infty} \beta_n e^{-n\xi}
\]

are two Entire Bicomplex Dirichlet series

\[
\Rightarrow \left\| \alpha_n \right\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \left\| \beta_n \right\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty
\]

i.e. given \( \varepsilon > 0 \) \( \exists m_1, m_2 \in \mathbb{N} \)

such that \( \left\| \alpha_n \right\|^{\frac{1}{n}} < \varepsilon \) \( \forall n \geq m_1 \) and \( \left\| \beta_n \right\|^{\frac{1}{n}} < \varepsilon \) \( \forall n \geq m_2 \)

Let \( m = \max (m_1, m_2) \)

Now \( \forall n \geq m, \left\| \alpha_n \right\|^{\frac{1}{n}} < \varepsilon \) and \( \left\| \beta_n \right\|^{\frac{1}{n}} < \varepsilon \)

Now,

\[
\Rightarrow \left\| \alpha_n \beta_n \right\|^{\frac{1}{n}} \leq \sqrt{2} \left( \left\| \alpha_n \right\|^{\frac{1}{n}} \right) \left( \left\| \beta_n \right\|^{\frac{1}{n}} \right) < (2)^{\frac{1}{2n}} \varepsilon^2
\]

\[
\Rightarrow \left\| \alpha_n \beta_n \right\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty
\]

Hence \( h(\xi) = \sum_{n=1}^{\infty} \left( \alpha_n \beta_n \right) e^{-n\xi} \) is an entire Bicomplex Dirichlet Series.

**Theorem 2.7:**

If \( f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi} \) and \( g(\xi) = \sum_{n=1}^{\infty} \beta_n e^{-n\xi} \) be two entire Bicomplex Dirichlet series, then the series \( \sum_{n=1}^{\infty} \left( \alpha_n \pm \beta_n \right) e^{-n\xi} \) is also an entire Bicomplex Dirichlet series.

**Theorem 2.8:**

If \( \sum_{n=1}^{\infty} \alpha_n e^{-n\xi} \) is an entire Bicomplex Dirichlet Series, then \( k^{th} \) derivative defined by \( \sum_{n=1}^{\infty} (-n)^k \alpha_n e^{-n\xi} \) is also an entire Bicomplex Dirichlet Series.
Proof:

\[ f(\xi) = \sum_{n=1}^{\infty} a_n \ e^{-n \xi} \] is an Entire Bicomplex Dirichlet Series

i.e. \[ \|a_n\|_{n} \to 0 \] as \( n \to \infty \)

Now,

\[ \|(-n)^k a_n\|_{n} = (n)^{-n} \|a_n\|_{n} \Rightarrow \] \[ (-n)^k a_n \] \[ \|n\] \[ \to 0 \] as \( n \to \infty \) \[ \therefore (n)^{-n} \to 0 \] as \( n \to \infty \)

**Theorem 2.9:**

If the Bicomplex Dirichlet series \( \sum_{n=1}^{\infty} a_n \ e^{-n \xi} \) is an entire Bicomplex Dirichlet series then the Bicomplex Dirichlet series

\[ \sum_{n=1}^{\infty} \frac{a_n}{(-n)^k} \ e^{-n \xi} \] obtained after \( k \)-times term-by-term integration of \( \sum_{n=1}^{\infty} a_n \ e^{-n \xi} \) is also an entire Bicomplex Dirichlet series.

**Proof:**

Let \( f(\xi) = \sum_{n=1}^{\infty} a_n \ e^{-n \xi} \) is an entire Bicomplex Dirichlet series i.e. \[ \|a_n\|_{n} \to 0 \] as \( n \to \infty \)

Now,

\[ \|a_n\|_{n^{k}} \to 0 \] as \( n \to \infty \) \[ \therefore \sum_{n=1}^{\infty} \frac{a_n}{(-n)^k} \ e^{-n \xi} \] is an entire Bicomplex Dirichlet series.

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**References and Notes**